# HOMEWORK 3 – HINTS/ANSWERS TO (MOST) PROBLEMS

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## Section 2.4: The precise definition of a limit

**2.4.2.** $\delta = 0.4$

**2.4.4.** $\delta = min\left\{1, \frac{1}{6}\right\} = \frac{1}{6}$ (there are many answers to this question, so don't worry if yours is different from mine)

**2.4.11. Note:** If this problem is too confusing for you, skip it! (it does more harm than good)

(a) $r = \sqrt{\frac{1000}{\pi}}$ (from now on, let's call this $a$)

(b) $|r - a| < min\left\{\frac{5}{\pi(2a+1)}, 1\right\} = \frac{5}{\pi(2a+1)}$ (for this, first set $|r - a| < 1$, solve for $r$ to get $a - 1 < r < a + 1$, then $|r + a| = r + a < 2a + 1$, so $\pi|r - a||r + a| < \pi|r - a|(2a + 1) < 5$, hence $|r - a| < \frac{5}{\pi(2a+1)}$

(c) $x = r =$ radius, $a = \sqrt{\frac{1000}{\pi}}$, $L = 1000$, $\epsilon = 5$, $\delta = min\left\{1, \frac{5}{\pi(2a+1)}\right\}$

**2.4.19.** See discussion section! This is an example of the 'easy case' with $\delta = \frac{3\epsilon}{4}$

**2.4.32.** This is an example of the 'complicated case' with $\delta = \min\left\{1, \frac{\epsilon}{19}\right\}$.

To get this $\delta$, notice that if $|x-2| < 1$, then $1 < x < 3$, and so $7 < x^2+2x+4 < 19$, so $|x^2 + 2x + 4| < 19$

**2.4.37.** This is again an example of the 'complicated case' with $\delta = min\left\{\frac{a}{2}, \epsilon\sqrt{a}\left(1 + \frac{1}{\sqrt{2}}\right)\right\}$

To get this $\delta$, notice that if $|x - a| < \frac{a}{2}$, then $\frac{a}{2} < x < \frac{3a}{2}$, and so in particular $\sqrt{x} + \sqrt{a} > \left(1 + \frac{1}{\sqrt{2}}\right)\sqrt{a}$ and then:

$$\frac{|x - a|}{\sqrt{x} + \sqrt{a}} < |x - a| \frac{1}{\left(1 + \frac{1}{\sqrt{2}}\right)\sqrt{a}} < \epsilon$$

which gives:

$$|x - a| < \epsilon\sqrt{a}\left(1 + \frac{1}{\sqrt{2}}\right)$$

The next two are optional, but good for practice:

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**2.4.42.** $\delta = \sqrt[4]{\frac{1}{M}}$

**2.4.43.** $\delta = e^M$ (where M is negative)

**2.4.44.** This is absolutely ridiculous, so feel free to skip it if you want!

(a) Let $M > 0$. We want to find $\delta$ such that if $|x-a| < \delta$, then $f(x)+g(x) > M$.

However, by replacing $M$ by $M + (1-c)$ in the definition of the limit of $f$, there exists $\delta_1$ such that if $|x-a| < \delta_1$, then $f(x) > M + (1-c)$

Moreover, by letting $\epsilon = 1$ in the definition of the limit of $g$, we know that there exits $\delta_2$ such that if $|x-a| < \delta_2$, then $|g(x) - c| < 1$. In particular, we get $g(x) - c > -1$, so $g(x) > c - 1$.

Hence, if you choose $\delta = min\{\delta_1, \delta_2\}$, then if $|x-a| < \delta$, we have $f(x) + g(x) > M + (1-c) + (c-1) = M$.

(b) Let $M > 0$. We want to find $\delta > 0$ such that if $|x-a| < \delta$, then $f(x)g(x) > M$.

By replacing $M$ by $M\left(\frac{2}{c}\right)$ in the definition of the limit of $f$, we know that there exists $\delta_1$ such that if $|x-a| < \delta_1$, then $f(x) > M\left(\frac{2}{c}\right)$

Moreover, by letting $\epsilon = \frac{c}{2} > 0$ in the definition of the limit of $g$, we know that there exists $\delta_2$ such that if $|x-a| < \delta_2$, then $|g(x) - c| < \frac{c}{2}$. In particular, $g(x) - c > -\frac{c}{2}$, so $g(x) > c - \frac{c}{2} = \frac{c}{2} > 0$.

Hence, if you choose $\delta = min\{\delta_1, \delta_2\}$, then if $|x-a| < \delta$, we have $f(x)g(x) > \left(M\frac{2}{c}\right)\left(\frac{c}{2}\right) = M$.

(c) Let $\boxed{M < 0}$. We want to find $\delta > 0$ such that if $|x-a| < \delta$, then $f(x)g(x) < M$.

By replacing $M$ by $M\left(\frac{2}{c}\right)$ in the definition of the limit of $f$, we know that there exists $\delta_1$ such that if $|x-a| < \delta_1$, then $f(x) > M\left(\frac{2}{c}\right)$

Moreover, by letting $\epsilon = -\frac{c}{2} > 0$ in the definition of the limit of $g$, we know that there exists $\delta_2$ such that if $|x-a| < \delta_2$, then $|g(x) - c| < -\frac{c}{2}$. In particular, $g(x) - c < -\frac{c}{2}$, so $g(x) < c - \frac{c}{2} = \frac{c}{2} < 0$.

Hence, if you choose $\delta = min\{\delta_1, \delta_2\}$, then if $|x-a| < \delta$, we have $f(x)g(x) < \left(M\frac{2}{c}\right)\left(\frac{c}{2}\right) = M$. (Note that in fact the $>$ in the first part, becomes a $<$ here precisely because $g(x) < 0$!)

## Section 2.5: Continuity

**2.5.3.** $-4$ ($f$ not defined at $-4$; neither), $-2$ (left-hand-side and right-hand-side limits not equal; continuous from the left), 2 (ditto; continuous from the right), 4 (left-hand-side limit does not exist; continuous from the right)

**2.5.11.** $f(2) = 4$ (You get this by solving for $f(2)$ in $3f(2) + f(2)g(2) = 36$)

**2.5.20.** Not continuous because the limit as $x \to 1$ equals $\frac{1}{2}$ (factor out the numerator and denominator), whereas $f(1) = 1$

**2.5.28.** Continuous because it's a ratio of two continuous functions (and the numerator/denominator are continuous because of composition of continuous functions), $Dom = \mathbb{R}$

**2.5.38.** $\tan^{-1}\left(\frac{2}{3}\right)$

**2.5.54.** Let $f(x) = \sin(x) - x^2 + x$

Then $f(1) = \sin(1) - 1 + 1 = \sin(1) > 0$, whereas $f(2) = \sin(2) - 4 + 2 = \sin(2) - 2 < 0$.

Moreover, $f$ is **continuous** on $[1, 2]$, hence by the **Intermediate Value Theorem** there exists one $c$ in $(1, 2)$ such that $f(c) = 0$, that is, $\sin(c) = c^2 - c$

**2.5.60.** Use the fact that $\sin(a + h) = \sin(a)\cos(h) + \sin(h)\cos(a)$

**2.5.69.** Define $f(t)$ to be the altitude of the monk on the first day, $g(t)$ to be the altitude of the monk on the second day, and let $h(t) = f(t) - g(t)$. Then $h(0) > 0$, $h(12) < 0$ (where 0 means $7AM$ and 12 means $12PM$), then by IVT, there is one number $c$ such that $h(c) = 0$, i.e. $f(c) = g(c)$

## Section 2.6: Limits at Infinity; Horizontal Asymptotes

**2.6.4.**

(a) 2
(b) $-1$
(c) $-\infty$
(d) $-\infty$
(e) $\infty$
(f) Horizontal asymptotes: $y = -1$, $y = 2$; Vertical asymptotes: $x = 0$, $x = 2$

**2.6.16.** 0 (factor out $x^3$ from the numerator and the denominator)

**2.6.26.** $-1$ (first factor out $x^2$ from the square root, and use $\sqrt{x^2} = |x| = -x$ because $x < 0$, then multiply by the conjugate form.

**2.6.38.** $\tan^{-1}(-\infty) = -\frac{\pi}{2}$ (by continuity of $\tan^{-1}$)

**2.6.43.** $y = 2$ (at $\pm\infty$), $x = 1$, $x = -2$ (factor out the denominator)

## Section 2.7: Derivatives and Rates of Change

**2.7.6.** $y = 9x - 15$

**2.7.12.**

(a) A runs with constant speed, while B is slow at first and then speeds up
(b) $\approx 8.5$ seconds
(c) 9 seconds

**2.7.17.** $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$

**2.7.18.** $y = 4x - 23$ ($y + 3 = 4(x - 5)$ is also acceptable)

**2.7.19.** $f(2) = 3$, $f'(2) = 4$